Exercise 43

Prove that $\lim_{x\to 0^+} \ln x = -\infty$.

Solution

According to Definition 4 and Definition 7, proving this infinite limit is logically equivalent to proving that

$$\text{if} \quad 0 < x < 0 + \delta \qquad \text{then} \qquad \ln x < N,$$

where N is any negative number. Start by working backwards, looking for a number δ that's greater than x.

$$\ln x < N$$

$$e^{\ln x} < e^N$$

$$x < e^N$$

Choose $\delta = e^N$. The hypothesis then becomes

$$0 < x < 0 + \delta$$

$$x < \delta$$

$$x < e^N$$

$$\ln x < \ln e^N$$

$$\ln x < N \ln e$$

$$\ln x < N.$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 0^+} \ln x = -\infty.$$